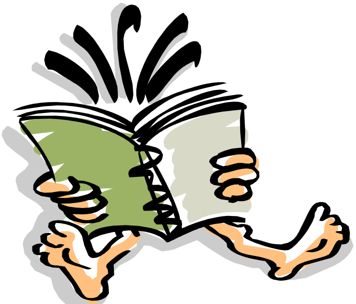
Asymptotic Notation & Master Method



Introduction to different cases of Analysis

- Worst case
  - Provides an upper bound on running time
  - An absolute guarantee that the algorithm would not run longer, no matter what the inputs are
- Best case
  - Provides a lower bound on running time
  - Input is the one for which the algorithm runs the fastest

$$Lower\ Bound \leq Running\ Time \leq Upper\ Bound$$

- Average case
  - Provides a prediction about the running time
  - Assumes that the input is random

Asymptotic Analysis

- To compare two algorithms with running times *f(n)* and *g(n),* we need a **rough measure** that characterizes **how fast each function grows.**
- *Hint:* use *rate of growth*
- Compare functions in the limit, that is, **asymptotically!**

  (i.e., for large values of *n*)

Rate of Growth

- Consider the example of buying *elephants* and *goldfish:*

  Cost: cost_of_elephants + cost_of_goldfish

  Cost ~ cost_of_elephants (approximation)

- The low order terms in a function are relatively insignificant for **large** *n*

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

i.e., we say that $n^4 + 100n^2 + 10n + 50$ and n^4 have the same **rate of growth**

Asymptotic Notation

- O notation: asymptotic "less than":
  - f(n)=O(g(n)) implies: f(n) "≤" g(n)
- Ω notation: asymptotic "greater than":
  - f(n)= Ω (g(n)) implies: f(n) "≥" g(n)
- Θ notation: asymptotic "equality":
  - f(n)= Θ (g(n)) implies: f(n) "=" g(n)

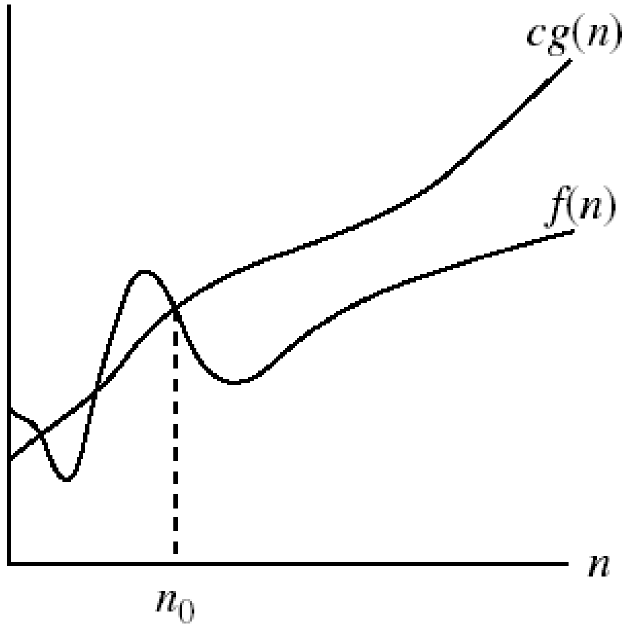
Big-O Notation

- We say $f_A(n)=30n+8$ is *order n*, or O (n)
  It is, at most, roughly *proportional* to *n*.
- $f_B(n)=n^2+1$ is *order* n^2, or $O(n^2)$. It is, at most, roughly proportional to n^2.
- In general, any $O(n^2)$ function is faster-growing than any $O(n)$ function.

Asymptotic notations

- *O-notation*

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} .$$



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Examples

- $2n^2 = O(n^3)$: $2n^2 \leq cn^3 \Rightarrow 2 \leq cn \Rightarrow c = 1$ and $n_0 = 2$

- $n^2 = O(n^2)$: $n^2 \leq cn^2 \Rightarrow c \geq 1 \Rightarrow c = 1$ and $n_0 = 1$

- $1000n^2+1000n = O(n^2)$:

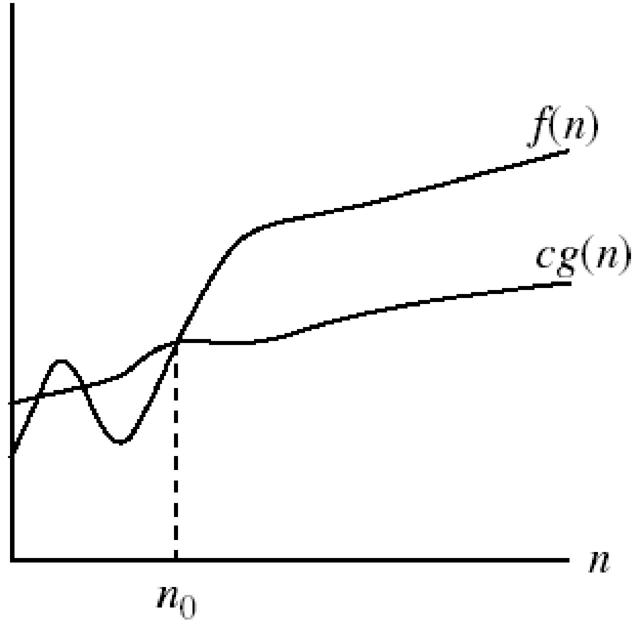
$1000n^2+1000n \leq 1000n^2+ n^2 = 1001n^2 \Rightarrow c=1001$ and $n_0 = 1000$

- $n = O(n^2)$: $n \leq cn^2 \Rightarrow cn \geq 1 \Rightarrow c = 1$ and $n_0 = 1$

Asymptotic notations (cont.)

- *Ω - notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



Ω(*g*(n)) is the set of functions with larger or same order of growth as *g*(n)

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples

- $5n^2 = \Omega(n)$

  $\exists\ c, n_0$ such that: $0 \le cn \le 5n^2 \Rightarrow cn \le 5n^2 \Rightarrow c = 1$ and $n_0 = 1$

- $100n + 5 \ne \Omega(n^2)$

  $\exists\ c, n_0$ such that: $0 \le cn^2 \le 100n + 5$

  $100n + 5 \le 100n + 5n\ (\forall\ n \ge 1) = 105n$

  $cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

  Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$

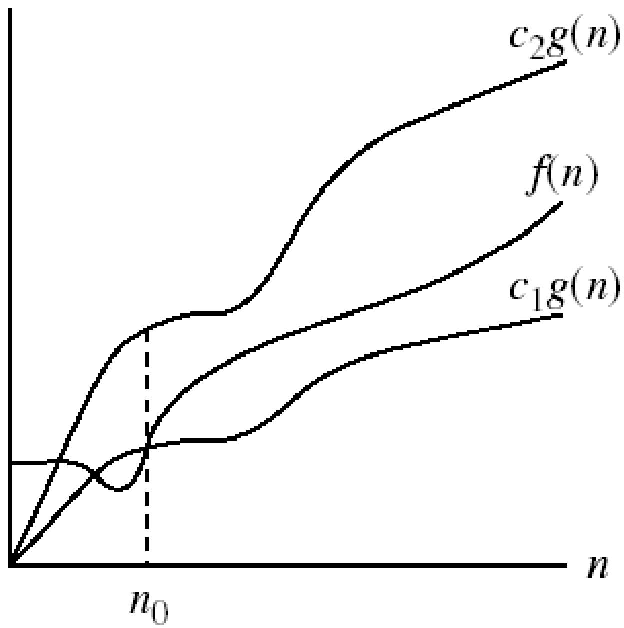
  $\Rightarrow$ contradiction: n cannot be smaller than a constant

- $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

Asymptotic notations (cont.)

- *Θ-notation*

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



Θ(*g*(n)) is the set of functions with the same order of growth as *g*(n)

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Examples

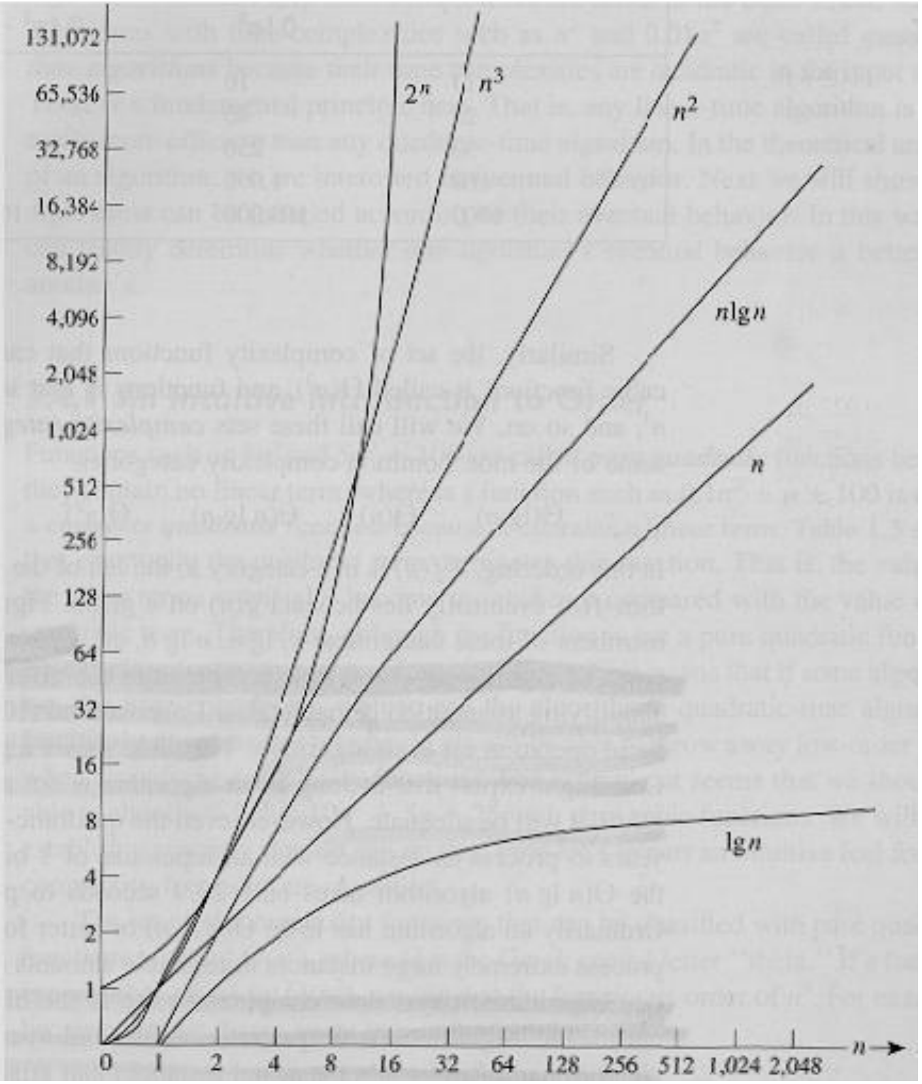
- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \leq \frac{1}{2} n^2 \; \forall n \geq 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \geq \frac{1}{2} n^2 - \frac{1}{2} n * \frac{1}{2} n \; (\forall n \geq 2) = \frac{1}{4} n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \neq \Theta(n^2)$: $c_1 n^2 \leq n \leq c_2 n^2$

  $\Rightarrow$ only holds for: $n \leq 1/c_1$

Common orders of magnitude



Master Method

- for $T(n) = aT(n/b)+f(n)$, n/b may be $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$.
  where $a \geq 1$, $b>1$ are positive integers, $f(n)$ be a non-negative function.

1. If $f(n)=O(n^{\log_b a-\varepsilon})$ for some $\varepsilon>0$, then $T(n)= \Theta(n^{\log_b a})$.
2. If $f(n)= \Theta(n^{\log_b a})$, then $T(n)= \Theta(n^{\log_b a} \lg n)$.
3. If $f(n)=\Omega(n^{\log_b a+\varepsilon})$ for some $\varepsilon>0$, and if $af(n/b) \leq cf(n)$ for some $c<1$ and all sufficiently large n, then $T(n)= \Theta(f(n))$.`

Implications of Master Theorem

- Comparison between $f(n)$ and $n^{\log_b a}$ (<,=,>)
- Must be asymptotically smaller (or larger) by a polynomial, i.e., n^{ε} for some $\varepsilon>0$.
- In case 3, the "regularity" must be satisfied, i.e., $af(n/b) \leq cf(n)$ for some $c<1$.
- There are gaps
  - between 1 and 2: $f(n)$ is smaller than $n^{\log_b a}$, but not polynomially smaller.
  - between 2 and 3: $f(n)$ is larger than $n^{\log_b a}$, but not polynomially larger.
  - in case 3, if the "regularity" fails to hold.

Application of Master Theorem

- $T(n) = 9T(n/3)+n;$
  - $a=9, b=3, f(n)=n$
  - $n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
  - $f(n)=O(n^{\log_3 9-\varepsilon})$ for $\varepsilon=1$
  - By case 1, $T(n) = \Theta(n^2)$.
- $T(n) = T(2n/3)+1$
  - $a=1, b=3/2, f(n)=1$
  - $n^{\log_b a} = n^{\log_{3/2} 1} = \Theta(n^0) = \Theta(1)$
  - By case 2, $T(n)= \Theta(\lg n)$.

Application of Master Theorem

- $T(n) = 3T(n/4)+n\lg n$;
  - a=3,b=4, $f(n) = n\lg n$
  - $n^{\log_b a} = n^{\log_4 3} = \Theta(n^{0.793})$
  - $f(n)= \Omega(n^{\log_4 3+\varepsilon})$ for $\varepsilon \approx 0.2$
  - Moreover, for large n, the “regularity” holds for c=3/4.
    - $af(n/b) = 3(n/4)\lg(n/4) \le (3/4)n\lg n = cf(n)$
  - By case 3, $T(n) = \Theta(f(n)) = \Theta(n\lg n)$.

Scope of research: To find the solution for Exception to Master Theorem

- $T(n) = 2T(n/2)+n\lg n$;
  - a=2,b=2, $f(n) = n\lg n$
  - $n^{\log_b a} = n^{\log_2 2} = \Theta(n)$
  - $f(n)$ is asymptotically larger than $n^{\log_b a}$, but not polynomially larger because
  - $f(n)/n^{\log_b a} = \lg n$, which is asymptotically less than n^{ε} for any $\varepsilon>0$.
  - Therefore,this is a gap between 2 and 3.

Assignment

- What is master method?
- How to find Big-O notation for given problem